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Optimal Harvesting in a Periodic Food Chain Model with Size Structures in Predators

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Abstract In this paper, we investigate a periodic food chain model with harvesting, where the predators have size structures and are described by first-order partial differential equations. First, we establish the existence of a unique non-negative solution by using the Banach fixed point theorem. Then, we provide optimality conditions by means of normal cone and adjoint system. Finally, we derive the existence of an optimal strategy by means of Ekeland's variational principle. Here the objective functional represents the net economic benefit yielded from harvesting.

Keywords Size-structure · Predator–prey model · Optimal harvesting

Mathematics Subject Classification 49K20 · 49K15 · 35F50 · 92D25

1 Introduction

Populations consist of individuals with many structural differences, which include age, (body) size, gender and genes. As a result, models with structures have been proposed and analyzed. During the past one hundred years, age-structured models have played significant role in the mathematical analysis and control of populations in biology and

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demography. To name a few, see [1–3,5,10,15,16,20,23,24,26] and the references therein. For example, Fister and Lenhart [10] investigated optimal harvesting control in a predator–prey model with age-structure in the prey.

For many populations especially in many ectothermic animal species, however, size of an individual has a strong influence upon dynamical processes like its feeding, growth and reproduction [25,27], which in turn affect the dynamics of the population as a whole. Here by size we mean some indices displaying the physiological or statistical characteristics of population individuals, such as mass, length, diameter, volume, and maturity. For example, for plants, an individual's size is important to capture light to grow. As a result, it is more realistic and natural to assume that vital rates such as mortality, fertility, and growth rates depend on size and time.

There have been many investigations on population models where the growth rate depends on the size. See, for example, [4,6–9,11–14,17–19,21,22] and the references therein. However, on the one hand, most of the studies only focus on a single species without stages [4,11,14,17–19] or with stages [12,22]. Only a few deal with interactions among species, particularly two-species predator–prey models [6,7,9,13,21]. Among them, Hallam and Henson [13] obtained the threshold on prey extinction, which is a function of size-dependent predation. The stability of equilibria is studied in [6,9]. Moreover, Bhattacharya and Martcheva [6] concluded that size-specific predation can destabilize a stable prey-only equilibrium. On the other hand, the literature on control problems in size-structured population models is scarce and most of the existing results are for single species. To the best of our knowledge, so far only Liu and He [21] studied the optimal harvesting of a two-species predator–prey model with size-structure in the prey. Moreover, natural populations are actually subject to seasonal fluctuations which have to be taken into account when the harvesting strategy is planned.

Motivated by the above discussion, in this paper, we consider a food chain of three species in a periodic environment with size structures in the predators and harvesting of all species. Here the top-level predator predates not only the low-level predator but also the prey. One of such examples consists of mouse, snake, and eagle.

The remaining part of this paper is organized as follows. First, we propose the model in Sect. 2. Then we study the existence of solution and continuous dependence of solutions on model parameters in Sects. 3 and 4, respectively. The last three sections are devoted to the optimal harvesting policy. Here the objective functional represents the net economic benefit yielded from harvesting. The adjoint system of the state system is derived in Sect. 5, followed by optimality conditions presented with a suitable normal cone in Sect. 6. Then the existence of a unique optimal policy is proved via Ekeland's variational principle in Sect. 7.

2 The Model

As mentioned earlier, in this paper, we study a food chain consists of two predators and one prey in a periodic environment. We assume that the top-level predator predates not only the low-level predator but also the prey. Moreover, there are size structures in and recruitments to the predators.

For simplicity, we assume that the two predators have the same maximal size and is l. Denote $R_+ = (0, \infty)$ and $Q = (0, l) \times R_+$. Let $p_1(x,t)$ and $p_2(x,t)$ represent the densities of the top-level predator and low-level predators of size x at time t, respectively, while $q(t)$ stands for the total number of the prey at time t. The model we consider is as follows,

$$\begin{cases} \frac{\partial p_1}{\partial t} + \frac{\partial(V_1(x,t)p_1)}{\partial x} = f_1(x,t) - \mu_1(x,t)p_1 - \alpha_1(x,t)p_1, & (x,t) \in Q, \\ \frac{\partial p_2}{\partial t} + \frac{\partial(V_2(x,t)p_2)}{\partial x} = f_2(x,t) - \mu_2(x,t)p_2 - \Phi_1(P_1(t))p_2 - \alpha_2(x,t)p_2, & \\ & (x,t) \in Q, \\ \frac{dq(t)}{dt} = g(t,q(t))q - \Phi_2(P_1(t))q - \Phi_3(P_2(t))q - \alpha_3(t)q, & t \in R_+, \\ V_1(0,t)p_1(0,t) = [f_3(P_2(t)) + f_4(q(t))] \int_0^l \beta_1(x)p_1(x,t)\,\mathrm{d}x, & t \in R_+, \\ V_2(0,t)p_2(0,t) = f_5(q(t)) \int_0^l \beta_2(x)p_2(x,t)\,\mathrm{d}x, & t \in R_+, \\ q(0) = q_0 > 0, \end{cases} \tag{2.1}$$

where $P_i(t) = \int_0^l p_i(x,t)\,\mathrm{d}x$ is the total number of the predator i at time t, $i = 1, 2$. Here the vital signs $V_1(x,t)$, $\mu_1(x,t)$, and $[f_3(P_2(t)) + f_4(q(t))]\beta_1(x)$ are, respectively, the growth rate, mortality, and fertility for the top-level predator; $V_2(x,t)$, $\mu_2(x,t)$, $f_5(q(t))\beta_2(x)$ are, respectively, the growth rate, mortality, and fertility for the low-level predator; $f_1(x,t)$ and $f_2(x,t)$ are respectively the recruitments of the two predators; $g(t,q(t))$ is the intrinsic growth rate of the prey; Φ_i's are the functional responses, $i = 1, 2, 3$. Since the food chain is in a periodic environment, we assume that

$$p_i(x,t) = p_i(x,t+T), \qquad (x,t) \in Q, \ \ i = 1, 2,$$

and

$$q(t) = q(t+T), \qquad t \in R_+,$$

where $T \in R_+$ is the period. The control variables $\alpha_1(x,t)$, $\alpha_2(x,t)$, $\alpha_3(t)$ are the harvesting efforts for the three populations, which belong to

$$\mathcal{U} = \left\{ (\alpha_1, \alpha_2, \alpha_3) \in L_T^\infty(Q) \times \in L_T^\infty(Q) \times L_T^\infty(R_+) \left| \begin{array}{l} 0 \le \alpha_i(x,t) \le N_i \\ \text{a.e. } (x,t) \in Q, i = 1, 2, \\ 0 \le \alpha_3(t) \le N_3 \\ \text{a.e. } t \in R_+ \end{array} \right. \right\},$$

where

$$L_T^\infty(Q) = \{h \in L^\infty(Q) : h(x,t) = h(x,t+T) \text{ a.e. } (x,t) \in Q\},$$
$$L_T^\infty(R_+) = \{h \in L^\infty(R_+) : h(t) = h(t+T) \text{ a.e. } t \in R_+\}.$$

Let $(p_1(x,t), p_2(x,t), q(t))$ be the solution of (2.1) corresponding to $(\alpha_1, \alpha_2, \alpha_3)$ $\in \mathcal{U}$. We investigate the following optimization problem,

$$\max_{(\alpha_1,\alpha_2,\alpha_3)\in\mathcal{U}} J(\alpha_1,\alpha_2,\alpha_3), \tag{2.2}$$

where

$$\begin{aligned} J(\alpha_1,\alpha_2,\alpha_3) = &\sum_{i=1}^{2}\int_0^T\int_0^l \omega_i(x,t)\alpha_i(x,t)p_i(x,t)\,\mathrm{d}x\mathrm{d}t \\ &-\frac{1}{2}\sum_{i=1}^{2}\int_0^T\int_0^l c_i\alpha_i^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &+\int_0^T \omega_3(t)\alpha_3(t)q(t)\,\mathrm{d}t - \frac{1}{2}\int_0^T c_3\alpha_3^2(t)\,\mathrm{d}t. \end{aligned}$$

Here the weight functions $\omega_1(\cdot,t)$, $\omega_2(\cdot,t)$, and $\omega_3(t)$ are respectively the economic values of the individuals of the three populations and all are T-periodic in t; c_1, c_2, and c_3 are the costs for harvesting. Therefore, $J(\alpha_1,\alpha_2,\alpha_3)$ represents the total net economic benefit yielded from harvesting during a time period of T.

We make the following assumptions throughout this paper.

(A1) For $i = 1$ and 2, $V_i : [0,l) \times R_+ \to R_+$ are bounded continuous functions, $V_i(x,t) > 0$ and $V_i(x,t) = V_i(x,t+T)$ for $(x,t) \in Q$, $V_i(l,t) = 0$ and $V_i(0,t) = 1$ for $t \in R_+$, and there are constants L_{V_i} such that

$$|V_i(x_1,t) - V_i(x_2,t)| \le L_{V_i}|x_1 - x_2| \qquad \text{for} t \in R_+ \text{ and } x_1, x_2 \in [0,l].$$

(A2) There exist $\overline{\beta}_i \in R_+$ such that $0 \le \beta_i(s) \le \bar{\beta}_i$ for $s \in (0,l)$, $i = 1, 2$.

(A3) $\begin{cases} \mu_1(x,t) = \mu_0(x) + \bar{\mu}_1(x,t) \text{ a.e. } (x,t) \in Q \text{ where } \mu_0 \in L^1_{loc}([0,l)) \\ \text{with } \mu_0(s) \ge 0 \text{ and } \int_0^l \mu_0(s)\,\mathrm{d}s = \infty,\ \bar{\mu}_1 \in L^\infty(Q) \\ \text{with } \bar{\mu}_1(x,t) \ge 0 \text{ and } \bar{\mu}_1(x,t) = \bar{\mu}_1(x,t+T) \text{ a.e. } (x,t) \in Q. \end{cases}$

(A4) $\mu_2 : [0,l) \times [0,T] \to R_+$ is a measurable function and $\mu_2(x,t) = \mu_2(x,t+T) \ge 0$ for all $(x,t) \in Q$.

(A5) $\mu_i(x,t) + V_{ix}(x,t) \ge 0$ for $(x,t) \in Q$, $i = 1, 2$.

(A6) There exists $\overline{B} > 0$ such that $0 \le g(t,S) = g(t+T,S) \le \bar{B}$ for $S, t \ge 0$ and g is Lipstchiz in the second variable, that is, there exists $L > 0$ such that

$$|g(t,S_1) - g(t,S_2)| \le L|S_1 - S_2| \qquad \text{for all } t, S_1, S_2 \ge 0.$$

(A7) There exist constants B_i and C_{Φ_i} $(i = 1, 2, 3)$ such that $0 \le \Phi_i(S) \le B_i$ for $S \ge 0$ and

$$|\Phi_i(S_1) - \Phi_i(S_2)| \le C_{\Phi_i}|S_1 - S_2| \qquad \text{for } S_1, S_2 \ge 0.$$

(A8) There exist constants C_i and L_i $(i = 3, 4, 5)$ such that $0 \le f_i(S) \le C_i$ for $S \ge 0$ and

$$|f_i(S_1) - f_i(S_2)| \le L_i|S_1 - S_2| \qquad \text{for } S_1, S_2 \ge 0.$$

(A9) $f_i \in L^\infty(Q)$ and $f_i(x,t) = f_i(x, t+T) \geq 0$ for $(x,t) \in Q$ and $i = 1, 2$.

Definition 2.1 For $i = 1, 2$, the unique solution $\varphi_i(t; t_0, x_{i0})$ of the initial-value problem $x'(t) = V_i(x,t)$ with $x(t_0) = x_{i0}$ is said to be a characteristic curve. Let $z_i(t) = \varphi_i(t; 0, 0)$ be the characteristic curve through $(0, 0)$ in the x-t plane.

Due to the periodicity of the state variables, $p_1(x,t)$, $p_2(x,t)$ and $q(t)$, we consider the case where $t > \max\{z_1^{-1}(l), z_2^{-1}(l)\} \triangleq \bar{t}$.

Definition 2.2 A three-tuple $(p_1(x,t), p_2(x,t), q(t)) \in L_T^\infty(Q) \times L_T^\infty(Q) \times L_T^\infty(R_+)$ of functions is a solution of (2.1) if it satisfies

$$\begin{aligned} p_1(x,t) &= p_1(0, t - z_1^{-1}(x))\Pi_1(x; x, t) \\ &\quad + \int_0^x \frac{f_1(r, \varphi_1^{-1}(r; t, x))}{V_1(r, \varphi_1^{-1}(r; t, x))} \frac{\Pi_1(x; x, t)}{\Pi_1(r; x, t)} \mathrm{d}r, \end{aligned} \tag{2.3}$$

$$\begin{aligned} p_2(x,t) &= p_2(0, t - z_2^{-1}(x))\Pi_2(x; x, t) \\ &\quad + \int_0^x \frac{f_2(r, \varphi_1^{-1}(r; t, x))}{V_2(r, \varphi_1^{-1}(r; t, x))} \frac{\Pi_2(x; x, t)}{\Pi_2(r; x, t)} \mathrm{d}r, \end{aligned} \tag{2.4}$$

$$q(t) = q_0 \exp\left\{ \int_0^t [g(\sigma, q(\sigma)) - \Phi_2(P_1(\sigma)) - \Phi_3(P_2(\sigma)) - \alpha_3(\sigma)]\mathrm{d}\sigma \right\}, \tag{2.5}$$

where

$$\begin{aligned} &\Pi_1(s; x, t) \\ &= \exp\left\{ -\int_0^s \frac{\mu_1(r, \varphi_1^{-1}(r; t, x)) + \alpha_1(r, \varphi_1^{-1}(r; t, x)) + V_{1x}(r, \varphi_1^{-1}(r; t, x))}{V_1(r, \varphi_1^{-1}(r; t, x))} \mathrm{d}r \right\}, \\ &\Pi_2(s; x, t) \\ &= \exp\left\{ -\int_0^s \frac{\mu_2(r, \varphi_2^{-1}(r; t, x)) + \Phi_1(P_1(t)) + \alpha_2(r, \varphi_2^{-1}(r; t, x)) + V_{2x}(r, \varphi_2^{-1}(r; t, x))}{V_2(r, \varphi_2^{-1}(r; t, x))} \mathrm{d}r \right\}. \end{aligned}$$

3 Well-Posedness of the State System

This section is devoted to the well-posedness of (2.1).

Let

$$M \triangleq \max \left\{ \begin{array}{l} q_0 e^{\bar{B}T}, C_5 \bar{\beta}_2 T \|f_2(\cdot,\cdot)\|_{L^1(Q)} e^{C_5 \bar{\beta}_2 T} + \|f_2(\cdot,\cdot)\|_{L^1(Q)}, \\ (C_3 + C_4)\bar{\beta}_1 T \|f_1(\cdot,\cdot)\|_{L^1(Q)} e^{(C_3+C_4)\bar{\beta}_1 T} + \|f_1(\cdot,\cdot)\|_{L^1(Q)} \end{array} \right\}.$$

Denote $\mathbf{X} = L^\infty(R_+, L^1(0,l)) \times L^\infty(R_+, L^1(0,l)) \times L^\infty(R_+)$. The solution space is defined as follows.

$$\mathcal{X} = \left\{ (p_1, p_2, q) \in \mathbf{X} \left| \begin{array}{l} 0 \leq q(t) \leq M \text{ a.e. } t \in R_+, \ p_i(x,t) \geq 0 \text{ a.e.} \\ (x,t) \in Q \text{ and } \int_0^l p_i(x,t)\,\mathrm{d}x \leq M, i = 1, 2 \end{array} \right. \right\}.$$

We define $G : \mathcal{X} \to \mathbf{X}$ by

$$(G(p_1, p_2, q)) = (G_1(p_1, p_2, q), G_2(p_1, p_2, q), G_3(p_1, p_2, q)),$$

where $G_1(p_1, p_2, q)$, $G_2(p_1, p_2, q)$, and $G(p_1, p_2, q)$ are defined respectively by the right hand sides of (2.3)–(2.5). Obviously, if $(p_1(x,t), p_2(x,t), q(t))$ is a fixed point of the map G then it is a solution of (2.1) and vice versa.

Theorem 3.1 *For a given $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{U}$, system* (2.1) *has a unique non-negative solution (p_1, p_2, q).*

Proof It suffices to show that G is a contraction mapping on $\mathcal{X}$.

First we show that G maps $\mathcal{X}$ into itself.

By assumption (A1), we have $V_1(0,t) = 1$ and $V_2(0,t) = 1$. Let $b_i(t) = p_i(0,t)$. Then, noting $\varphi_i^{-1}(0; t, x) = t - z_i^{-1}(x)$ $(i = 1, 2)$, we have

$$\begin{aligned}
&b_1(t)\\
&\quad = V_1(0,t) p_1(0,t)\\
&\quad = [f_3(P_2(t)) + f_4(q(t))] \int_0^l \beta_1(x) p_1(x,t) \mathrm{d}x\\
&\quad \le [C_3 + C_4] \int_0^l \beta_1(x) \left[p_1(0, \varphi_1^{-1}(0; t, s)) + \int_0^x \frac{f_1(r, \varphi_1^{-1}(r; t, x))}{V_1(r, \varphi_1^{-1}(r; t, x))} \mathrm{d}r \mathrm{d}x \right]\\
&\quad \le [C_3 + C_4] \left[\int_0^l \beta_1(x) b_1(\varphi_1^{-1}(0; t, x)) \mathrm{d}x \right.\\
&\qquad\qquad \left. + \int_0^l \beta_1(x) \int_0^x \frac{f_1(r, \varphi_1^{-1}(r; t, x))}{V_1(r, \varphi_1^{-1}(r; t, x))} \mathrm{d}r \mathrm{d}x \right]\\
&\quad = [C_3 + C_4] \left[\int_0^l \beta_1(x) b_1(t - z_1^{-1}(x)) \mathrm{d}x \right.\\
&\qquad\qquad \left. + \int_0^l \beta_1(x) \int_{\varphi_1^{-1}(0; t, x)}^t f_1(\varphi_1(\sigma; t, x), \sigma) \mathrm{d}\sigma \mathrm{d}x \right]\\
&\quad \le [C_3 + C_4] \bar{\beta}_1 \int_0^l b_1(t - z_1^{-1}(x)) \mathrm{d}x\\
&\qquad + [C_3 + C_4] \bar{\beta}_1 \int_0^l \int_0^t f_1(\varphi_1(\sigma; t, x), \sigma) \mathrm{d}\sigma \mathrm{d}x\\
&\quad \le [C_3 + C_4] \bar{\beta}_1 \int_0^t b_1(\sigma) \mathrm{d}\sigma + [C_3 + C_4] \bar{\beta}_1 \| f_1(\cdot, \cdot) \|_{L^1(Q)}.
\end{aligned}$$

(Here we have used the transformation $\sigma = \varphi_1^{-1}(r; t, x)$ to obtain the last equality. By Definition 2.1, $\sigma = t$ when $r = x$ while $\sigma = \varphi_1^{-1}(0; t, x)$ when $r = 0$. Moreover, it follows from $r = \varphi_1(\sigma; t, x)$ that $dr = \frac{\mathrm{d}\varphi_1(\sigma; t, x)}{\mathrm{d}\sigma} \mathrm{d}\sigma = V_1(\varphi_1(\sigma; t, x), \sigma) \mathrm{d}\sigma =$

$V_1(r, \varphi_1^{-1}(r; t, x))\mathrm{d}\sigma$. The same transformation and a similar one $s = \varphi_2(r; t, x)$ will be used in the coming discussion.) Similarly, we have

$$b_2(t) \le C_5\bar{\beta}_2 \int_0^t b_2(\sigma)\mathrm{d}\sigma + C_5\bar{\beta}_2 \| f_2(\cdot, \cdot)\|_{L^1(Q)}.$$

Due to the periodicity of $b_1(t)$ and $b_2(t)$, we consider the case where $t \in [\bar{t}, \bar{t} + T]$ only. It follows from Bellman's lemma that

$$b_1(t) \le [C_3 + C_4]\bar{\beta}_1 \| f_1(\cdot, \cdot)\|_{L^1(Q)} e^{[C_3+C_4]\bar{\beta}_1 T}$$

and

$$b_2(t) \le C_5\bar{\beta}_2 \| f_2(\cdot, \cdot)\|_{L^1(Q)} e^{C_5\bar{\beta}_2 T}.$$

Now, we consider $G(p_1, p_2, q) = (G_1(p_1, p_2, q), G_2(p_1, p_2, q), G_3(p_1, p_2, q))$. We can see that

$$\begin{aligned}
&\int_0^l |G_1(p_1, p_2, q)|(x, t)\mathrm{d}x \\
&\le \int_0^l p_1(0, \varphi_1^{-1}(0; t, x))\Pi_1(x; x, t)\mathrm{d}x \\
&\quad + \int_0^l \int_0^x \frac{f_1(r, \varphi_1^{-1}(r; t, x))}{V_1(r, \varphi_1^{-1}(r; t, x))} \frac{\Pi_1(x; x, t)}{\Pi_1(r; x, t)} \mathrm{d}r\mathrm{d}x \\
&\le \int_0^l p_1(0, \varphi_1^{-1}(0; t, x))\mathrm{d}x + \int_0^l \int_0^x \frac{f_1(r, \varphi_1^{-1}(r; t, x))}{V_1(r, \varphi_1^{-1}(r; t, x))}\mathrm{d}r\mathrm{d}x \\
&= \int_0^l b_1(t - z_1^{-1}(x))\mathrm{d}x + \int_0^l \int_{\varphi_1^{-1}(0;t,x)}^t f_1(\varphi_1(\sigma; t, x), \sigma)\mathrm{d}\sigma\mathrm{d}x \\
&\le \int_0^l b_1(t - z_1^{-1}(x))\mathrm{d}x + \int_0^l \int_0^t f_1(\varphi_1(\sigma; t, x), \sigma)\mathrm{d}\sigma\mathrm{d}x \\
&\le \int_0^t b_1(\sigma)\mathrm{d}\sigma + \| f_1(\cdot, \cdot)\|_{L^1(Q)} \\
&\le [C_3 + C_4]\bar{\beta}_1 T \| f_1(\cdot, \cdot)\|_{L^1(Q)} e^{[C_3+C_4]\bar{\beta}_1 T} + \| f_1(\cdot, \cdot)\|_{L^1(Q)}.
\end{aligned}$$

Similarly,

$$\int_0^l |G_2(p_1, p_2, q)|(x, t)\mathrm{d}x \le C_5\bar{\beta}_2 T \| f_2(\cdot, \cdot)\|_{L^1(Q)} e^{C_5\bar{\beta}_2 T} + \| f_2(\cdot, \cdot)\|_{L^1(Q)}$$

and

$$|G_3(p_1, p_2, q)|(t) \le q_0 e^{\bar{B}T}.$$

It follows that G is a mapping from $\mathcal{X}$ to $\mathcal{X}$.

Next, we discuss the compressibility of the mapping G.

By (2.3), we can get

$$
\begin{aligned}
&\int_0^l |G_1(p_1,p_2,q)-G_1(p_1',p_2',q')|(x,t)\mathrm{d}x\\
&=\int_0^l \mid p_1(0,\varphi_1^{-1}(0;t,x))\Pi_1(x;x,t)-p_1'(0,\varphi_1^{-1}(0;t,x))\Pi_1(x;x,t)\mid \mathrm{d}x\\
&\le \int_0^l |p_1(0,\varphi_1^{-1}(0;t,x))-p_1'(0,\varphi_1^{-1}(0;t,x))|\mathrm{d}x\\
&=\int_0^l |b_1(t-z_1^{-1}(x))-b_1'(t-z_1^{-1}(x))|\mathrm{d}x\\
&\le \int_0^t |b_1(\sigma)-b_1'(\sigma)|\mathrm{d}\sigma\\
&=\int_0^t \Big|[f_3(P_2(\sigma))+f_4(q(\sigma))]\int_0^l \beta_1(x)p_1(x,\sigma)\mathrm{d}x\\
&\quad -[f_3(P_2'(\sigma))+f_4(q'(\sigma))]\int_0^l \beta_1(x)p_1'(x,\sigma)\mathrm{d}x\Big|\mathrm{d}\sigma\\
&\le \bar{\beta}_1(C_3+C_4)\int_0^t\int_0^l |p_1(x,\sigma)-p_1'(x,\sigma)|\mathrm{d}x\mathrm{d}\sigma\\
&\quad +\bar{\beta}_1(L_3+L_4)\int_0^t \big[|P_2(\sigma)-P_2'(\sigma)|+|(q(\sigma))-q'(\sigma)|\big]\int_0^l p_1'(x,\sigma)\mathrm{d}x\mathrm{d}\sigma\\
&\le \bar{\beta}_1(L_3+L_4)M\int_0^t\left[\int_0^l |p_2(x,\sigma)-p_2'(x,\sigma)|\mathrm{d}x+|(q(\sigma))-q'(\sigma)|\right]\mathrm{d}\sigma\\
&\quad +\bar{\beta}_1(C_3+C_4)\int_0^t\int_0^l |p_1(x,\sigma)-p_1'(x,\sigma)|\mathrm{d}x\mathrm{d}\sigma\\
&\le M_1\int_0^t\left[|(q(\sigma))-q'(\sigma)|+\int_0^l |p_1(x,\sigma)-p_1'(x,\sigma)|\mathrm{d}x\right.\\
&\quad \left.+\int_0^l |p_2(x,\sigma)-p_2'(x,\sigma)|\mathrm{d}x\right]\mathrm{d}\sigma,
\end{aligned}
$$

where $M_1=\max\{\bar{\beta}_1(C_3+C_4),\bar{\beta}_1(L_3+L_4)M\}$.

By (2.4), we obtain

$$
\begin{aligned}
G_2(p_1,p_2,q)(x,t)&=p_2(0,\varphi_2^{-1}(0;t,x))\Pi_2(x;x,t)\\
&\quad +\int_0^x \frac{f_2(r,\varphi_2^{-1}(r;t,x))}{V_2(r,\varphi_2^{-1}(r;t,x))}\frac{\Pi_2(x;x,t)}{\Pi_2(r;x,t)}\mathrm{d}r.
\end{aligned}
$$

With the integral transform $s=\varphi_2^{-1}(r;t,x)$, the above equality can be turned into

$$G_2(p_1,p_2,q)(x,t)=b_2(\varphi_2^{-1}(0;t,x))E(\varphi_2^{-1}(0;t,x);x,t,P_1(t))\\ +\int_{\varphi_2^{-1}(0;t,x)}^{t} f_2(\varphi_2(s;t,x),s)E(s;x,t,P_1(t))\mathrm{d}s,$$

where

$$E(r;x,t,P_1(t))=\exp\left\{-\int_r^t \mu_2(\varphi_2(s;t,x),s)+\Phi_1(P_1(t))\right.\\ \left.+\alpha_2(\varphi_2(s;t,x),s)+V_{2x}(\varphi_2(s;t,x),s)\mathrm{d}s\right\}.$$

Let

$$M_2=C_{\Phi_1}T(M+l\|f(\cdot,\cdot)\|_{L^1(Q)}),\\ M_3=\max\left\{L_5\bar{\beta}_2M+M_2M_1,\,M_2M_1,\,C_5\bar{\beta}_2+M_2M_1\right\}.$$

Then we have

$$\begin{aligned}
&\int_0^l |G_2(p_1,p_2,q)-G_2(p_1',p_2',q')|(x,t)\mathrm{d}x\\
&=\int_0^l |b_2(\varphi_2^{-1}(0;t,x))E(\varphi_2^{-1}(0;t,x);x,t,P_1(t))\\
&\quad+\int_{\varphi_2^{-1}(0;t,x)}^{t} f_2(\varphi_2(s;t,x),s)E(s;x,t,P_1(t))\mathrm{d}s\\
&\quad-b_2'(\varphi_2^{-1}(0;t,x))E(\varphi_2^{-1}(0;t,x);x,t,P_1'(t))\\
&\quad+\int_{\varphi_2^{-1}(0;t,x)}^{t} f_2(\varphi_2(s;t,x),s)E(s;x,t,P_1'(t))\mathrm{d}s|\mathrm{d}x\\
&\le\int_0^l b_2'(\varphi_2^{-1}(0;t,x))\int_{\varphi_2^{-1}(0;t,x)}^{t}|\Phi_1(P_1(t))-\Phi_1(P_1'(t))|\mathrm{d}s\mathrm{d}x\\
&\quad+\int_0^l |b_2(\varphi_2^{-1}(0;t,x))-b_2'(\varphi_2^{-1}(0;t,x))|\mathrm{d}x\\
&\quad+\int_0^l\int_{\varphi_2^{-1}(0;t,x)}^{t} f_2(\varphi_2(s;t,x),s)\int_s^t|\Phi_1(P_1(t))-\Phi_1(P_1'(t))|\mathrm{d}r\mathrm{d}s\mathrm{d}x\\
&\le\int_0^t |b_2(\sigma)-b_2'(\sigma)|\mathrm{d}\sigma+C_{\Phi_1}MT\int_0^l|p_1(x,t)-p_1'(x,t)|\mathrm{d}x\\
&\quad+\|f(\cdot,\cdot)\|_{L^1(Q)}C_{\Phi_1}lT\int_0^l|p_1(x,t)-p_1'(x,t)|\mathrm{d}x\\
&\le\int_0^t |f_5(q(\sigma))\int_0^l\beta_2(x)p_2(x,\sigma)\mathrm{d}x-f_5(q'(\sigma))\int_0^l\beta_2(x)p_2'(x,\sigma)\mathrm{d}x|\mathrm{d}\sigma
\end{aligned}$$

$$
\begin{aligned}
&+[C_{\Phi_1}T(M+l\|f(\cdot,\cdot)\|_{L^1(Q)})]\int_0^l |p_1(x,t)-p_1'(x,t)|\mathrm{d}x\\
&\le \int_0^t \left[|f_5(q(\sigma))-f_5(q'(\sigma))|\int_0^l \beta_2(s)p_2(x,\sigma)\mathrm{d}x\right]\mathrm{d}\sigma\\
&+\int_0^t |f_5(q'(\sigma))|\int_0^l \beta_2(x)|p_2(x,\sigma)-p_2'(x,\sigma)|\mathrm{d}x\mathrm{d}\sigma\\
&+[C_{\Phi_1}T(M+l\|f(\cdot,\cdot)\|_{L^1(Q)})]\int_0^l |p_1(x,t)-p_1'(x,t)|\mathrm{d}x\\
&\le L_5\bar{\beta}_2 M\int_0^t |q(\sigma)-q'(\sigma)|\mathrm{d}\sigma + C_5\bar{\beta}_2\int_0^t\int_0^l |p_2(x,\sigma)-p_2'(x,\sigma)|\mathrm{d}x\mathrm{d}\sigma\\
&+M_2M_1\int_0^t\left[|(q(\sigma))-q'(\sigma)|+\int_0^l |p_1(x,\sigma)-p_1'(x,\sigma)|\mathrm{d}x\right.\\
&\left.+\int_0^l |p_2(x,\sigma)-p_2'(x,\sigma)|\mathrm{d}x\right]\mathrm{d}\sigma\\
&=(L_5\bar{\beta}_2M+M_2M_1)\int_0^t |(q(\sigma))-q'(\sigma)|\mathrm{d}\sigma\\
&+M_2M_1\int_0^t\int_0^l |p_1(x,\sigma)-p_1'(x,\sigma)|\mathrm{d}x\mathrm{d}\sigma\\
&+(C_5\bar{\beta}_2+M_2M_1)\int_0^t\int_0^l |p_2(x,\sigma)-p_2'(x,\sigma)|\mathrm{d}x\mathrm{d}\sigma\\
&\le M_3\int_0^t\left[|(q(\sigma))-q'(\sigma)|+\int_0^l |p_1(x,\sigma)-p_1'(x,\sigma)|\mathrm{d}x\right.\\
&\left.+\int_0^l |p_2(x,\sigma)-p_2'(x,\sigma)|\mathrm{d}x\right]\mathrm{d}\sigma,
\end{aligned}
$$

By (2.5), we have

$$
\begin{aligned}
&|G_3(p_1,p_2,q)-G_3(p_1',p_2',q')|(t)\\
&=q_0\left|\exp\left\{\int_0^t g(\sigma,q(\sigma))-\Phi_2(P_1(\sigma))-\Phi_3(P_2(\sigma))-\alpha_3(\sigma)\mathrm{d}\sigma\right\}\right.\\
&\left.-\exp\left\{\int_0^t g(\sigma,q'(\sigma))-\Phi_2(P_1'(\sigma))-\Phi_3(P_2'(\sigma))-\alpha_3(\sigma)\mathrm{d}x\sigma\right\}\right|\\
&\le q_0e^{2T\bar{B}}\int_0^t [|g(\sigma,q(\sigma))-g(\sigma,q'(\sigma))|+|\Phi_2(P_1(\sigma))-\Phi_2(P_1'(\sigma))|\\
&+|\Phi_3(P_2(\sigma))-\Phi_3(P_2'(\sigma))|]\mathrm{d}\sigma\\
&\le q_0e^{2T\bar{B}}\int_0^t [L|q(\sigma)-q'(\sigma)|+C_{\Phi_2}\int_0^l |p_1(x,\sigma)-p_1'(x,\sigma)|\mathrm{d}x\\
&+C_{\Phi_3}\int_0^l |p_2(x,\sigma)-p_2'(x,\sigma)|\mathrm{d}x]\mathrm{d}\sigma
\end{aligned}
$$

$$
\begin{aligned}
&\leq M_4 \int_0^t \left[|(q(\sigma)) - q'(\sigma)| + \int_0^l |p_1(x,\sigma) - p_1'(x,\sigma)| \mathrm{d}x \right. \\
&\quad \left. + \int_0^l |p_2(x,\sigma) - p_2'(x,\sigma)| \mathrm{d}x \right] \mathrm{d}\sigma,
\end{aligned}
$$

where $M_4 = q_0 e^{2T\bar{B}} \max\{L, C_{\Phi_2}, C_{\Phi_3}\}$.

Now we use the Banach fixed point theorem to demonstrate that the mapping G has only one fixed point. Due to the periodicity of the elements in the set $\mathcal{X}$, we consider the case $t \in [\bar{t}, \bar{t}+T]$ only. Define a new norm by

$$
\|(p_1, p_2, q)\|_* = Ess \sup_{t\in[\bar{t},\bar{t}+T]} \left\{ e^{-\lambda t} [|q(t)| + \int_0^l |p_1(x,t)| \mathrm{d}x + \int_0^l |p_2(x,t)| \mathrm{d}x] \right\}
$$

for any $(p_1, p_2, q) \in \mathcal{X}$ and some $\lambda > 0$, which is equivalent to the usual norm. Then we have

$$
\begin{aligned}
&\|G(p_1, p_2, q) - G(p_1', p_2', q')\|_* \\
&\quad = \|(G_1(p_1, p_2, q) - G_1(p_1', p_2', q'), G_2(p_1, p_2, q) - G_2(p_1', p_2', q'), \\
&\qquad G_3(p_1, p_2, q) - G_3(p_1', p_2', q'))\|_* \\
&\quad = Ess \sup_{t\in[\bar{t},\bar{t}+T]} \left\{ e^{-\lambda t} \left[|G_3(p_1, p_2, q) - G_3(p_1', p_2', q')| \right.\right. \\
&\qquad + \int_0^l |G_1(p_1, p_2, q) - G_1(p_1', p_2', q')| \mathrm{d}x \\
&\qquad \left.\left. + \int_0^l |G_2(p_1, p_2, q) - G_2(p_1', p_2', q')| \mathrm{d}x \right] \right\} \\
&\quad \leq (M_1 + M_3 + M_4) Ess \sup_{t\in[\bar{t},\bar{t}+T]} \left\{ e^{-\lambda t} \left[\int_0^t (|(q(\sigma)) - q'(\sigma)| \right.\right. \\
&\qquad \left.\left. + \int_0^l |p_1(x,\sigma) - p_1'(x,\sigma)| \mathrm{d}x + \int_0^l |p_2(x,\sigma) - p_2'(x,\sigma)| \mathrm{d}x) \mathrm{d}\sigma \right] \right\} \\
&\quad \doteq M_5 Ess \sup_{t\in[\bar{t},\bar{t}+T]} \left\{ e^{-\lambda t} \int_0^t e^{\lambda\sigma} \left[e^{-\lambda\sigma} (|(q_1(\sigma)) - q_2(\sigma)| \right.\right. \\
&\qquad \left.\left. + \int_0^l |p_1(x,\sigma) - p_1'(x,\sigma)| \mathrm{d}x + \int_0^l |p_2(x,\sigma) - p_2'(x,\sigma)| \mathrm{d}x) \right] \mathrm{d}\sigma \right\} \\
&\quad \leq \frac{M_5}{\lambda} \|(p_1 - p_1', p_2 - p_2', q - q')\|_*.
\end{aligned}
$$

Choose λ such that $\lambda > M_5$. Then G becomes a contraction on the space of $(\mathcal{X}, \|\cdot\|_*)$. By the Banach fixed point theorem, G owns a unique fixed point, which is the solution of (2.1). □

4 The Continuous Dependence of Solutions

In this section, we will discuss the continuous dependence of solutions on the control variable.

Theorem 4.1 *For any* $(\alpha_1, \alpha_2, \alpha_3)$, $(\alpha_1', \alpha_2', \alpha_3') \in \mathcal{U}$, *let* (p_1, p_2, q) *and* (p_1', p_2', q') *be solutions of* (2.1) *corresponding to* $(\alpha_1, \alpha_2, \alpha_3)$ *and* $(\alpha_1', \alpha_2', \alpha_3')$, *respectively. Then there exist positive constants* M_{14} *and* M_{15} *such that*

$$
\begin{aligned}
&\|p_1 - p_1'\|_{L^\infty(0,T;L^1(0,l))} + \|p_2 - p_2'\|_{L^\infty(0,T;L^1(0,l))} + \|q - q'\|_{L^\infty(0,T)} \\
&\le M_{14}T(\|\alpha_1 - \alpha_1'\|_{L^\infty(0,T;L^1(0,l))} + \|\alpha_2 - \alpha_2'\|_{L^\infty(0,T;L^1(0,l))} + \|\alpha_3 - \alpha_3'\|_{L^\infty(0,T)})
\end{aligned}
$$

and

$$
\begin{aligned}
&\|p_1 - p_1'\|_{L^1(Q_T)} + \|p_2 - p_2'\|_{L^1(Q_T)} + \|q - q'\|_{L^1(0,T)} \\
&\le M_{15}T\left(\|\alpha_1 - \alpha_1'\|_{L^1(Q_T)} + \|\alpha_2 - \alpha_2'\|_{L^1(Q_T)} + \|\alpha_3 - \alpha_3'\|_{L^1(0,T)}\right).
\end{aligned}
$$

Proof We only prove the first estimate as the proof for the second one is similar. Since (p_1, p_2, q) and (p_1', p_2', q') are solutions of (2.1) corresponding to $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_1', \alpha_2', \alpha_3')$, respectively, we have

$$
\begin{aligned}
&|q(t) - q'(t)| \\
&= q_0\Big|\exp\left\{\int_0^t g(\sigma, q(\sigma)) - \Phi_2(P_1(\sigma)) - \Phi_3(P_2(\sigma)) - \alpha_3(\sigma)\mathrm{d}\sigma\right\} \\
&\quad - \exp\left\{\int_0^t g(\sigma, q'(\sigma)) - \Phi_2(P_1'(\sigma)) - \Phi_3(P_2'(\sigma)) - \alpha_3'(\sigma)\mathrm{d}\sigma\right\}\Big| \\
&\le q_0 e^{2T\bar{B}}\Big|\int_0^t \Big[-g(\sigma, q(\sigma)) - \Phi_2(P_1(\sigma)) - \Phi_3(P_2(x,\sigma)) - \alpha_3(\sigma) \\
&\quad + g(\sigma, q'(\sigma)) + \Phi_2(P_1'(\sigma)) + \Phi_3(P_2'(\sigma)) + \alpha_3'(\sigma)\Big]\mathrm{d}\sigma\Big| \\
&= q_0 e^{2T\bar{B}}\Big|\int_0^t \Big[-g(\sigma, q(\sigma)) - \Phi_2\left(\int_0^l p_1(x,\sigma)\mathrm{d}x\right) - \Phi_3\left(\int_0^l p_2(x,\sigma)\mathrm{d}x\right) - \alpha_3(\sigma) \\
&\quad + g(\sigma, q'(\sigma)) + \Phi_2\left(\int_0^l p_1'(x,\sigma)\mathrm{d}x\right) + \Phi_3\left(\int_0^l p_2'(x,\sigma)\mathrm{d}x\right) + \alpha_3'(\sigma)\Big]\mathrm{d}\sigma\Big| \\
&\le q_0 e^{2T\bar{B}}\int_0^t \Big[|g(\sigma, q(\sigma)) - g(\sigma, q'(\sigma))| \\
&\quad + \left|\Phi_2\left(\int_0^l p_1(x,\sigma)\mathrm{d}x\right) - \Phi_2\left(\int_0^l p_1'(x,\sigma)\mathrm{d}x\right)\right| \\
&\quad + \left|\Phi_3\left(\int_0^l p_2(x,\sigma)\mathrm{d}x\right) - \Phi_3\left(\int_0^l p_2'(x,\sigma)\mathrm{d}x\right)\right| + |\alpha_3(\sigma) - \alpha_3'(\sigma)|\Big]\mathrm{d}\sigma \\
&\le q_0 e^{2T\bar{B}}L_1\int_0^t |q(\sigma) - q'(\sigma)|\mathrm{d}\sigma \\
&\quad + C_{\Phi_2}q_0 e^{2T\bar{B}}\int_0^t\int_0^l |p_1(x,\sigma) - p_1'(x,\sigma)|\mathrm{d}x\mathrm{d}\sigma
\end{aligned}
$$

$$+ C_{\Phi_3} q_0 e^{2T\bar{B}} \int_0^t \int_0^l |p_2(x,\sigma) - p_2'(x,\sigma)| \mathrm{d}x \mathrm{d}\sigma$$

$$+ q_0 e^{2T\bar{B}} \int_0^t |\alpha_3(\sigma) - \alpha_3'(\sigma)| \mathrm{d}\sigma,$$

$$\int_0^l |p_1(x,t) - p_1'(x,t)| \mathrm{d}x$$

$$= \int_0^l \Big| b_1(\varphi_1^{-1}(0;t,x)) \Pi_1(x;x,t) + \int_0^x \frac{f_1(r,\varphi_1^{-1}(r;t,x))}{V_1(r,\varphi_1^{-1}(r;t,x))} \frac{\Pi_1(x;x,t)}{\Pi_1(r;t,x)} \mathrm{d}r$$

$$- b_1'(\varphi_1^{-1}(0;t,x)) \Pi_1'(x;x,t) - \int_0^x \frac{f_1(r,\varphi_1^{-1}(r;t,x))}{V_1(r,\varphi_1^{-1}(r;t,x))} \frac{\Pi_1'(x;x,t)}{\Pi_1'(r;x,t)} \mathrm{d}r \Big| \mathrm{d}x$$

$$\le \int_0^l |b_1(t - z_1^{-1}(x)) - b_1'(t - z_1^{-1}(x))| \mathrm{d}x$$

$$+ (C_3 + C_4) M \bar{\beta}_1 \int_0^l \int_0^t |\alpha_1(\varphi_1(\sigma;t,x),\sigma) - \alpha_1'(\varphi_1(\sigma;t,x),\sigma)| \mathrm{d}\sigma \mathrm{d}x$$

$$+ \int_0^l \int_0^t f_1(\varphi(s;t,x),s) \int_s^t |\alpha_1(\varphi_1(\sigma;t,x),\sigma) - \alpha_1'(\varphi_1(\sigma;t,x),\sigma)| \mathrm{d}\sigma \mathrm{d}s \mathrm{d}x$$

$$\le M_6 \int_0^t |q(\sigma) - q'(\sigma)| \mathrm{d}\sigma + M_7 \int_0^t \int_0^l |p_1(x,\sigma) - p_1'(x,\sigma)| \mathrm{d}x \mathrm{d}\sigma$$

$$+ M_8 \int_0^t \int_0^l |p_2(x,\sigma) - p_2'(x,\sigma) \mathrm{d}x \mathrm{d}\sigma$$

$$+ M_9 \int_0^t \int_0^l |\alpha_1(\varphi_1(\sigma;t,x),\sigma) - \alpha_1'(\varphi_2(\sigma;t,x),\sigma)| \mathrm{d}\sigma \mathrm{d}x,$$

$$\int_0^l |p_2(x,t) - p_2'(x,t)| \, \mathrm{d}x$$

$$= \int_0^l \Big| b_2(\varphi_2^{-1}(0;t,x)) \Pi_2(x;x,t) + \int_0^x \frac{f_2(r,\varphi_2^{-1}(r;t,x))}{V_2(r,\varphi_2^{-1}(r;t,x))} \frac{\Pi_2(x;x,t)}{\Pi_2(r;x,t)} \, \mathrm{d}r$$

$$- b_2'(\varphi_2^{-1}(0;t,x)) \Pi_2'(x;x,t) - \int_0^x \frac{f_2(r,\varphi_2^{-1}(r;t,x))}{V_2(r,\varphi_2^{-1}(r;t,x))} \frac{\Pi_2'(x;x,t)}{\Pi_2'(r;x,t)} \, \mathrm{d}r \Big| \, \mathrm{d}x$$

$$\le \int_0^l |b_2(\varphi_2^{-1}(0;t,x)) - b_2'(\varphi_2^{-1}(0;t,x)) \Big| \, \mathrm{d}x$$

$$+ \int_0^l b_2'(\varphi_2^{-1}(0;t,x)) \int_0^x \frac{|\alpha_2(r,\varphi_2^{-1}(r;t,x)) - \alpha_2'(r,\varphi_2^{-1}(r;t,x))|}{V_2(r,\varphi_2^{-1}(r;t,x))} \, \mathrm{d}r \, \mathrm{d}x$$

$$+ \int_0^x \frac{|\Phi_1(P_1(t)) - \Phi_1(P_1'(t))|}{V_2(r,\varphi_2^{-1}(r;t,x))} \, \mathrm{d}r \, \mathrm{d}x + \int_0^l \int_0^x \frac{f_2(r,\varphi_2^{-1}(r;t,x))}{V_2(r,\varphi_2^{-1}(r;t,x))}$$

$$\times \int_r^x \frac{|\alpha_2(\sigma,\varphi_2^{-1}(\sigma;t,x)) - \alpha_2'(\sigma,\varphi_2^{-1}(\sigma;t,x))|}{V_2(\sigma,\varphi_2^{-1}(\sigma;t,x))} \, \mathrm{d}\sigma \, \mathrm{d}r \, \mathrm{d}x$$

$$+ \int_r^x \frac{|\Phi_1(P_1(t)) - \Phi_1(P_1'(t))|}{V_2(\sigma,\varphi_2^{-1}(\sigma;t,x))} \, \mathrm{d}\sigma \, \mathrm{d}r \, \mathrm{d}x$$

$$\le M_{10} \int_0^t |q(\sigma) - q'(\sigma)| \, \mathrm{d}\sigma + M_{11} \int_0^t \int_0^l |p_1(x,\sigma) - p_1'(x,\sigma)| \, \mathrm{d}x \, \mathrm{d}\sigma$$

$$+M_{12}\int_0^t\int_0^l |p_2(x,\sigma)-p_2'(x,\sigma)|\,\mathrm{d}x\,\mathrm{d}\sigma$$
$$+M_{13}\int_0^t\int_0^l |\alpha_2(\varphi_2(\sigma;t,x),\sigma)-\alpha_2'(\varphi_2(\sigma;t,x),\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}x.$$

Here

$$\Pi_1'(s;x,t) = \exp\left\{-\int_0^s \frac{\mu_1(r,\varphi_1^{-1}(r;t,x))+\alpha_1'(r,\varphi_1^{-1}(r;t,x))+V_{1x}(r,\varphi_1^{-1}(r;t,x))}{V_1(r,\varphi_1^{-1}(r;t,x))}\,\mathrm{d}r\right\},$$
$$\Pi_2'(s;x,t) = \exp\left\{-\int_0^s \frac{\mu_2(r,\varphi_2^{-1}(r;t,x))+\Phi_1(P_1'(t))+\alpha_2'(r,\varphi_2^{-1}(r;t,x))+V_{2x}(r,\varphi_2^{-1}(r;t,x))}{V_2(r,\varphi_2^{-1}(r;t,x))}\,\mathrm{d}r\right\}.$$

Now the result follows immediately from the above analysis. □

5 The Adjoint System

In this section, we will derive the adjoint system of (2.1), which is necessary for optimality and the existence of a unique optimal policy.

Lemma 5.1 *Let* (p_1^*, p_2^*, q^*) *be the solution of* (2.1) *corresponding to* $(\alpha_1^*,\alpha_2^*,\alpha_3^*)\in\mathcal{U}$. *For each* $(v_1,v_2,v_3)\in\mathcal{T}_{\mathcal{U}}(\alpha_1^*,\alpha_2^*,\alpha_3^*)$ *such that* $(\alpha_1^*+\varepsilon v_1,\alpha_2^*+\varepsilon v_2,\alpha_3^*+\varepsilon v_3)\in\mathcal{U}$ *for sufficiently small* $\varepsilon>0$, *we have*

$$\frac{1}{\varepsilon}[p_1^\varepsilon-p_1^*]\to z_1,\qquad \frac{1}{\varepsilon}[p_2^\varepsilon-p_2^*]\to z_2,\qquad \frac{1}{\varepsilon}[q^\varepsilon-q^*]\to z_3$$

as $\varepsilon\to 0$, *where* $(p_1^\varepsilon,p_2^\varepsilon,q^\varepsilon)$ *is the solution of* (2.1) *corresponding to* $(\alpha_1^*+\varepsilon v_1,\alpha_2^*+\varepsilon v_2,\alpha_3^*+\varepsilon v_3)$ *and* (z_1,z_2,z_3) *is the solution of the following system*

$$\begin{cases}\frac{\partial z_1}{\partial t}+\frac{\partial(V_1(x,t)z_1)}{\partial s}=-[\mu_1(x,t)+\alpha_1^*(x,t)]z_1(x,t)-v_1(x,t)p_1^*(x,t),\\ \frac{\partial z_2}{\partial t}+\frac{\partial(V_2(x,t)z_2)}{\partial s}=-[\mu_2(x,t)+\Phi_1(P_1^*(t))+\alpha_2^*(x,t)]z_2(x,t)\\ \qquad\qquad -[v_2(x,t)+\Phi_1'(P_1^*(t))Z_1(t)]p_2^*(x,t),\\ \frac{dz_3}{dt}=-[\Phi_2'(P_1^*(t))Z_1(t)+\Phi_3'(P_2^*(t))Z_2(t)+v_3(t)]q^*(t)\\ \qquad +[g(t,q^*(t))+q^*(t)\frac{\partial g(t,q^*(t))}{\partial q}-\alpha_3^*(t)-\Phi_2(P_1^*(t))-\Phi_3(P_2^*(t))]z_3(t),\\ z_1(0,t)=[f_3(P_2^*(t))+f_4(q^*(t))]\int_0^l\beta_1(x)z_1(x,t)\,\mathrm{d}x\\ \qquad +[f_3'(P_2^*(t))Z_2(t)+f_4'(q^*(t))z_3(t)]\int_0^l\beta_1(x)p_1^*(x,t)\,\mathrm{d}x\\ z_2(0,t)=f_5(q^*(t))\int_0^l\beta_2(x)z_2(x,t)\,\mathrm{d}x+f_5'(q^*(t))z_3(t)\int_0^l\beta_2(x)p_2^*(x,t)\,\mathrm{d}x,\end{cases}\tag{5.1}$$

where

$$z_1(x,t)=z_1(x,t+T),\quad z_2(x,t)=z_2(x,t+T),\quad z_3(t)=z_3(t+T),$$
$$Z_1(t)=\int_0^l z_1(x,t)\,\mathrm{d}x,\qquad Z_2(t)=\int_0^l z_2(x,t)\,\mathrm{d}x,$$
$$P_1^*(t)=\int_0^l p_1^*(x,t)\,\mathrm{d}x,\qquad P_2^*(t)=\int_0^l p_2^*(x,t)\,\mathrm{d}x.$$

Proof The existence and uniqueness of solution to (5.1) can be established by a similar way as that in the proof of Theorem 3.1. According to Lemma 3.1.3 in [23], $\lim_{\varepsilon\to 0}\frac{p_i^\varepsilon - p_i^*}{\varepsilon}$ $(i=1,2)$ and $\lim_{\varepsilon\to 0}\frac{q^\varepsilon - q^*}{\varepsilon}$ make sense. Since (p_1^*, p_2^*, q^*) and $(p_1^\varepsilon, p_2^\varepsilon, q^\varepsilon)$ are solutions of (2.1) corresponding to $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$ and $(\alpha_1^*+\varepsilon v_1, \alpha_2^*+\varepsilon v_2, \alpha_3^*+\varepsilon v_3)$, respectively, it follows that $\frac{1}{\varepsilon}[p_1^\varepsilon - p_1^*]$, $\frac{1}{\varepsilon}[p_2^\varepsilon - p_2^*]$, and $\frac{1}{\varepsilon}[q^\varepsilon - q^*]$ must be solutions of

$$\begin{cases} \frac{\partial(\frac{1}{\varepsilon}(p_1^\varepsilon - p_1^*))}{\partial t} + \frac{\partial(\frac{1}{\varepsilon}V_1(p_1^\varepsilon - p_1^*))}{\partial x} = -(\mu_1+\alpha_1^*)[\frac{1}{\varepsilon}(p_1^\varepsilon - p_1^*)] - v_1 p_1^\varepsilon, \\ \frac{\partial(\frac{1}{\varepsilon}(p_2^\varepsilon - p_2^*))}{\partial t} + \frac{\partial(\frac{1}{\varepsilon}(p_2^\varepsilon - p_2^*))}{\partial x} = -(\mu_2+\alpha_2^*)[\frac{1}{\varepsilon}(p_2^\varepsilon - p_2^*)] - v_2 p_2^\varepsilon \\ \qquad -\frac{1}{\varepsilon}[\Phi_1(P_1^\varepsilon(t))p_2^\varepsilon - \Phi_1(P_1^*(t))p_2^*], \\ \frac{d(\frac{1}{\varepsilon}(q^\varepsilon - q^*))}{dt} = \frac{1}{\varepsilon}[g(t,q^\varepsilon(t))q^\varepsilon(t) - g(t,q^*(t))q^*(t)] - \alpha_3^*[\frac{1}{\varepsilon}(q^\varepsilon - q^*)] \\ \qquad -v_3 q^\varepsilon - \frac{1}{\varepsilon}[\Phi_2(P_1^\varepsilon(t))q^\varepsilon(t) - \Phi_2(P_1^*(t))q^*(t)] \\ \qquad -\frac{1}{\varepsilon}[\Phi_3(P_2^\varepsilon(t))q^\varepsilon(t) - \Phi_3(P_2^*(t))q^*(t)], \\ \frac{1}{\varepsilon}(p_1^\varepsilon - p_1^*)(0,t) = \frac{1}{\varepsilon}[f_3(P_2^\varepsilon(t)) + f_4(q^\varepsilon(t))]\int_0^l \beta_1(x)p_1^\varepsilon(x,t)\,\mathrm{d}x \\ \qquad -\frac{1}{\varepsilon}[f_3(P_2^*(t)) + f_4(q^*(t))]\int_0^l \beta_1(x)p^*(x,t)\,\mathrm{d}x, \\ \frac{1}{\varepsilon}(p_2^\varepsilon - p_2^*)(0,t) = \frac{1}{\varepsilon}[f_5(q^\varepsilon(t))\int_0^l \beta_2(x)p_2^\varepsilon(x,t)\,\mathrm{d}x \\ \qquad - f_5(q^*(t))\int_0^l \beta_2(x)p_2^*(x,t)\,\mathrm{d}x], \end{cases} \tag{5.2}$$

where

$$\begin{aligned} \frac{1}{\varepsilon}(p_i^\varepsilon - p_i^*)(x,t) &= \frac{1}{\varepsilon}(p_i^\varepsilon - p_i^*)(x,t+T) \qquad (i=1,2), \\ \frac{1}{\varepsilon}(q^\varepsilon - q^*)(t) &= \frac{1}{\varepsilon}(q^\varepsilon - q^*)(t+T), \\ \frac{1}{\varepsilon}(P_i^\varepsilon - P_i^*)(t) &= \int_0^l \frac{1}{\varepsilon}(p_i^\varepsilon - p_i^*)(x,t)\,\mathrm{d}x \qquad (i=1,2), \\ P_i^*(t) &= \int_0^l p_i^*(x,t)\,\mathrm{d}x \qquad (i=1,2). \end{aligned}$$

□

It follows from Theorem 4.1 that

$$\begin{aligned} &\frac{1}{\varepsilon}[g(t,q^\varepsilon(t))q^\varepsilon(t) - g(t,q^*(t))q^*(t)] \\ &\quad = \frac{1}{\varepsilon}[g(t,q^\varepsilon(t)) - g(t,q^*(t))]q^\varepsilon(t) + g(t,q^*(t))\frac{1}{\varepsilon}[q^\varepsilon - q^*] \\ &\quad \to g(t,q^*(t))z_3(t) + q^*(t)\frac{\partial g(t,q^*(t))}{\partial q}z_3(t) \end{aligned} \tag{5.3}$$

as $\varepsilon \to 0$. Similarly, we have

$$\frac{1}{\varepsilon}[\Phi_1(P_1^\varepsilon(t))p_2^\varepsilon(x,t) - \Phi_1(P_1^*(t))p_2^*(x,t)]$$

$$\to \Phi_1(P_1^*(t))z_2(x,t) + \Phi_1'(P_1^*(t))Z_1(t)p_2^*(x,t), \tag{5.4}$$

$$\frac{1}{\varepsilon}[\Phi_2(P_1^\varepsilon(t))q^\varepsilon(t) - \Phi_2(P_1^*(t))q^*(t)]$$
$$\to \Phi_2(P_1^*(t))z_3(t) + \Phi_2'(P_1^*(t))Z_1(t)q^*(t), \tag{5.5}$$

$$\frac{1}{\varepsilon}[\Phi_3(P_2^\varepsilon(t))q^\varepsilon(t) - \Phi_3(P_2^*(t))q^*(t)]$$
$$\to \Phi_3(P_2^*(t))z_3(t) + \Phi_3'(P_2^*(t))Z_2(t)q^*(t), \tag{5.6}$$

$$\frac{1}{\varepsilon}\left[f_3(P_2^\varepsilon(t))\int_0^l \beta_1(s)p_1^\varepsilon(x,t)\,\mathrm{d}x - f_3(P_2^*(t))\int_0^l \beta_1(x)p_1^*(x,t)\,\mathrm{d}x\right]$$
$$\to f_3(P_2^*(t))\int_0^l \beta_1(x)z_1(x,t)\,\mathrm{d}x + f_3'(P_2^*(t))Z_2(t)\int_0^l \beta_1(x)p_1^*(x,t)\,\mathrm{d}x, \tag{5.7}$$

$$\frac{1}{\varepsilon}\left[f_4(q^\varepsilon(t))\int_0^l \beta_1(x)p_1^\varepsilon(x,t)\,\mathrm{d}x - f_4(q^*(t))\int_0^l \beta_1(x)p_1^*(x,t)\,\mathrm{d}x\right]$$
$$\to f_4(q^*(t))\int_0^l \beta(x)_1z_1(x,t)\,\mathrm{d}x + f_4'(q^*(t))z_3(t)\int_0^l \beta_1(x)p_1^*(x,t)\,\mathrm{d}x, \tag{5.8}$$

$$\frac{1}{\varepsilon}\left[f_5(q^\varepsilon(t))\int_0^l \beta_2(x)p_2^\varepsilon(x,t)\,\mathrm{d}x - f_5(q^*(t))\int_0^l \beta_2(x)p_2^*(x,t)\,\mathrm{d}x\right]$$
$$\to f_5(q^*(t))\int_0^l \beta_2(x)z_2(x,t)\,\mathrm{d}x + f_5'(q^*(t))z_3(t)\int_0^l \beta_2(s)p_2^*(x,t)\,\mathrm{d}x \tag{5.9}$$

as $\varepsilon \to 0$. Taking $\varepsilon \to 0$ in (5.2) and using (5.3)–(5.9) yield the required result.

Next we consider the following adjoint system of (2.1),

$$\begin{cases} \frac{\partial \xi_1}{\partial t} + V_1(x,t)\frac{\partial \xi_1}{\partial x} = [\mu_1(x,t) + \alpha_1^*(x,t)]\xi_1 + \omega_1(x,t)\alpha_1^*(x,t) \\ \qquad + \eta(t)q^*(t)\Phi_2'(P_1^*(t)) + \Phi_1'(P_1^*(t))\int_0^l p_2^*(x,t)\xi_2(x,t)\,\mathrm{d}x, \\ \qquad - \xi_1(0,t)\beta_1(x)[f_3(P_2^*(t)) + f_4(q^*(t))] \\ \frac{\partial \xi_2}{\partial t} + V_2(x,t)\frac{\partial \xi_2}{\partial x} = [\mu_2(x,t) + \alpha_2^*(x,t) + \Phi_1(P^*(t))]\xi_2 + \omega_2(x,t)\alpha_2^*(x,t) \\ \qquad + \eta(t)q^*(t)\Phi_3'(P_2^*(t)) - \xi_2(0,t)\beta_2(x)f_5(q^*(t)), \\ \qquad - f_3'(P_2^*(t))\xi_1(0,t)\int_0^l \beta_1(x)p_1^*(x,t)\,\mathrm{d}x, \\ \frac{d\eta}{dt} = -[g(t,q^*(t)) + q^*(t)\frac{\partial g(t,q^*(t))}{\partial q} - \Phi_2(P_1^*(t)) - \Phi_3(P_2^*(t)) - \alpha_3^*(t)]\eta \\ \quad + \omega_3(t)\alpha_3^*(t) - \xi_1(0,t)f_4'(q^*(t))\int_0^l \beta_1(x)p_1^*(x,t)\,\mathrm{d}x \\ \quad - \xi_2(0,t)f_5'(q^*(t))\int_0^l \beta_2(x)p_2^*(x,t)\,\mathrm{d}x, \end{cases} \tag{5.10}$$

where

$$\xi_i(x,t) = \xi_i(x,t+T) \qquad (i=1,2),$$
$$\eta(t) = \eta(t+T),$$
$$\xi_i(l,t) = 0 \qquad (i=1,2),$$
$$P_i^*(t) = \int_0^l p_i^*(x,t)\,\mathrm{d}x \qquad (i=1,2).$$

Treating (5.10) in the same manner as that in Theorem 4.1, we can get the following result.

Theorem 5.2 *For each* $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{U}$*, the adjoint system* (5.10) *has a unique bounded solution* (ξ_1, ξ_2, η)*. Moreover, there exists a positive constant* B *such that*

$$\|\xi_1 - \xi_1'\|_{L^\infty(Q_T} + \|\xi_2 - \xi_2'\|_{L^\infty(Q_T} + \|\eta - \eta'\|_{L^\infty(0,T)}$$
$$\leq BT(\|\alpha_1 - \alpha_1'\|_{L^\infty(Q_T)} + (\|\alpha_2 - \alpha_2'\|_{L^\infty(Q_T)} + \|\alpha_3 - \alpha_3'\|_{L^\infty(0,T)}),$$

where (ξ_1, ξ_2, η) *and* (ξ_1', ξ_2', η') *are solutions of the adjoint system* (5.10) *corresponding to* $(\alpha_1, \alpha_2, \alpha_3)$ *and* $(\alpha_1', \alpha_2', \alpha_3')$*, respectively.*

6 Optimality Conditions

Theorem 6.1 *Let* (p_1^*, p_2^*, q^*) *be the solution of* (2.1) *corresponding to the optimal policy* $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$*. Then*

$$\alpha_1^*(x,t) = \mathcal{F}_1\left[\frac{[\omega_1(x,t) + \xi_1(x,t)]p_1^*(x,t)}{c_1}\right], \tag{6.1}$$

$$\alpha_2^*(x,t) = \mathcal{F}_2\left[\frac{[\omega_2(x,t) + \xi_2(x,t)]p_2^*(x,t)}{c_2}\right], \tag{6.2}$$

$$\alpha_3^*(t) = \mathcal{F}_3\left[\frac{[\omega_3(t) + \eta(t)]q^*(t)}{c_3}\right], \tag{6.3}$$

where the truncated mappings $\mathcal{F}_i$ *are given by*

$$(\mathcal{F}_1 h)(x,t) = \begin{cases} 0, & h(x,t) < 0, \\ h(x,t), & 0 \leq h(s,t) \leq N_1, \\ N_1, & h(x,t) > N_1, \end{cases} \tag{6.4}$$

$$(\mathcal{F}_2 h)(x,t) = \begin{cases} 0, & h(x,t) < 0, \\ h(x,t), & 0 \leq h(x,t) \leq N_2, \\ N_2, & h(x,t) > N_2, \end{cases} \tag{6.5}$$

$$(\mathcal{F}_3 h)(t) = \begin{cases} 0, & h(t) < 0, \\ h(t), & 0 \leq h(t) \leq N_3, \\ N_3, & h(t) > N_3, \end{cases} \tag{6.6}$$

where $(\xi_1(s,t), \xi_2(x,t), \eta(t))$ *is the solution of the adjoint system* (5.10).

Proof For any element of the tangent cone $(v_1, v_2, v_3) \in \mathcal{T}_{\mathcal{U}}(\alpha_1^*, \alpha_2^*, \alpha_3^*)$, we have $(\alpha_1^* + \varepsilon v_1, \alpha_2^* + \varepsilon v_2, \alpha_3^* + \varepsilon v_3) \in \mathcal{U}$ for sufficiently small $\varepsilon > 0$. Let $(p_1^\varepsilon, p_2^\varepsilon, q^\varepsilon)$ be the solution of (2.1) corresponding to $(\alpha_1^* + \varepsilon v_1, \alpha_2^* + \varepsilon v_2, \alpha_3^* + \varepsilon v_3)$. From the optimality of $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$, we get

$$\sum_{i=1}^{2}\int_0^T\int_0^l \omega_i[\alpha_i^* + \varepsilon v_i]p_i^\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \sum_{i=1}^{2}\frac{1}{2}\int_0^T\int_0^l c_i[\alpha_i^* + \varepsilon v_i]^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T \omega_3[\alpha_3^* + \varepsilon v_3]q^\varepsilon \,\mathrm{d}t - \frac{1}{2}\int_0^T c_3[\alpha_3^* + \varepsilon v_3]^2 \,\mathrm{d}t$$
$$\le \sum_{i=1}^{2}\int_0^T\int_0^l \omega_i\alpha_i^* p_i^* \,\mathrm{d}x\,\mathrm{d}t - \sum_{i=1}^{2}\frac{1}{2}\int_0^T\int_0^l c_i[\alpha_i^*]^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T \omega_3\alpha_3^* q^* \,\mathrm{d}t - \frac{1}{2}\int_0^T c_3[\alpha_3^*]^2 \,\mathrm{d}t.$$

Consequently,

$$0 \ge \sum_{i=1}^{2}\int_0^T\int_0^l \left(\omega_i\alpha_i^*\frac{p_i^\varepsilon - p_i^*}{\varepsilon} + \omega_i v_i p_i^\varepsilon - c_i\alpha_i^* v_i - \frac{1}{2}\varepsilon c_i v_i^2\right) \mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T \left(\omega_3\alpha_3^*\frac{q^\varepsilon - q^*}{\varepsilon} + \omega_3 v_3 q^\varepsilon - c_3\alpha_3^* v_3 - \frac{1}{2}\varepsilon c_3 v_3^2\right) \mathrm{d}t.$$

By Lemma 5.1, as $\varepsilon \to 0^+$, we have

$$0 \ge \sum_{i=1}^{2}\int_0^T\int_0^l (\omega_i\alpha_i^* z_i)(x,t)\,\mathrm{d}x\,\mathrm{d}t + \sum_{i=1}^{2}\int_0^T\int_0^l [(\omega_i p_i^* - c_i\alpha_i^*)v_i](x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T (\omega_3\alpha_3^* z)(t)\,\mathrm{d}t + \int_0^T [(\omega_3 q^* - c_3\alpha_3^* v_3](t)\,\mathrm{d}t.$$

Multiplying (5.1) by $\xi_1(s,t)$, $\xi_2(x,t)$, and $\eta(t)$ and then integrating on Q_T and $[0,T]$ and using (5.10), we obtain

$$\sum_{i=1}^{2}\int_0^T\int_0^l (\omega_i\alpha_i^* z_i)(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T (\omega_3\alpha_3^* z_3)(t)\,\mathrm{d}t$$
$$= \sum_{i=1}^{2}\int_0^T\int_0^l (\xi_i p_i^* v_i)(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T (\eta q^* v_3)(t)\,\mathrm{d}t.$$

Then

$$0 \ge \sum_{i=1}^{2}\int_0^T\int_0^l \{[\omega_i(x,t) + \xi_i(x,t)]p_i^*(x,t) - c_i\alpha_i^*(x,t)\}v_i(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T \{[\omega_3(t) + \eta(t)]q^*(t) - c_3\alpha_3^*(t)\}v_3(t)\,\mathrm{d}t$$

for each $(v_1, v_2, v_3) \in \mathcal{T}_{\mathcal{U}}(\alpha_1^*, \alpha_2^*, \alpha_3^*)$. Thus $([(\omega_1 + \xi_1)p_1^* - c_1\alpha_1^*](s,t), [(\omega_2 + \xi_2)p_2^* - c_2\alpha_2^*](x,t), [(\omega_3 + \eta)q^* - c_3\alpha_3^*](t)) \in \mathcal{N}_{\mathcal{U}}(\alpha_1^*, \alpha_2^*, \alpha_3^*)$, which implies the conclusion of this theorem. □

7 Existence of a Unique Optimal Policy

To apply the Ekeland's variational principle, we need the following mapping,

$$\tilde{J}(\alpha_1, \alpha_2, \alpha_3) = \begin{cases} J(\alpha_1, \alpha_2, \alpha_3), & (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{U}, \\ -\infty, & \text{otherwise}. \end{cases}$$

Lemma 7.1 *The functional $\tilde{J}(\alpha_1, \alpha_2, \alpha_3)$ is upper semi-continuous.*

Proof Suppose that $(\alpha_1^n, \alpha_2^n, \alpha_3^n) \to (\alpha_1, \alpha_2, \alpha_3)$ as $n \to +\infty$. Without loss of generality, we assume that $(\alpha_1^n, \alpha_2^n, \alpha_3^n) \in \mathcal{U}$ for all n. Let (p_{1n}, p_{2n}, q_n) and (p_1, p_2, q) be solutions of (2.1) corresponding to $(\alpha_1^n, \alpha_2^n, \alpha_3^n)$ and $(\alpha_1, \alpha_2, \alpha_3)$, respectively.

By Theorem 4.1, we know that, for any $t \in (0, T)$,

$$p_{1n}(\cdot, t) \to p_1(\cdot, t), \qquad p_{2n}(\cdot, t) \to p_2(\cdot, t), \qquad q_n(t) \to q(t)$$

as $n \to +\infty$. By the Riesz theorem, there exists a subsequence (still denoted by $(\alpha_1^n, \alpha_2^n, \alpha_3^n)$) such that, for any $(s,t) \in Q_T$, $(x,t) \in Q_T$, and $t \in [0, T]$,

$$\begin{aligned} \alpha_1^n(s,t) &\to \alpha_1(s,t), & \alpha_2^n(x,t) &\to \alpha_2(x,t), & \alpha_3^n(t) &\to \alpha_3(t), \\ p_{1n}(s,t) &\to p_1(s,t), & p_{2n}(x,t) &\to p_2(x,t), & q_n(t) &\to q(t), \end{aligned} \tag{7.1}$$

$$(\alpha_1^n(s,t))^2 \to (\alpha_1(s,t))^2, \quad (\alpha_2^n(x,t))^2 \to (\alpha_2(x,t))^2, \quad (\alpha_3^n(t))^2 \to (\alpha_3(t))^2 \tag{7.2}$$

as $n \to +\infty$. By (7.1), it's easy for us to get

$$\begin{aligned} \omega_1(s,t)\alpha_1^n(s,t)p_{1n}(s,t) &\to \omega_1(s,t)\alpha_1(s,t)p_1(s,t), \\ \omega_2(x,t)\alpha_2^n(x,t)p_{2n}(x,t) &\to \omega_2(x,t)\alpha_2(x,t)p_2(x,t), \\ \omega_3(t)\alpha_3^n(t)q_n(t) &\to \omega_3(t)\alpha_3(t)q(t) \end{aligned}$$

as $n \to +\infty$. From (7.2), using the Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{n\to+\infty} \int_0^T \int_0^l (\alpha_1^n(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_0^l (\alpha_1(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t, \\ \lim_{n\to+\infty} \int_0^T \int_0^l (\alpha_2^n(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_0^l (\alpha_2(x,t))^2 \,\mathrm{d}x\,\mathrm{d}t, \\ \lim_{n\to+\infty} \int_0^T (\alpha_3^n(t))^2 \,\mathrm{d}t &= \int_0^T (\alpha_3(t))^2 \,\mathrm{d}t. \end{aligned}$$

It follows from Fatou's lemma that

$$\lim_{n\to+\infty} \sup_{(\alpha_1^n,\alpha_2^n,\alpha_3^n)\in\mathcal{U}} \tilde{J}(\alpha_1^n,\alpha_2^n,\alpha_3^n) \le \tilde{J}(\alpha_1,\alpha_2,\alpha_3),$$

which shows that $\tilde{J}(\alpha_1,\alpha_2,\alpha_3)$ is upper semi-continuous. □

Theorem 7.2 *If* $B_1 T(\frac{1}{c_1}+\frac{1}{c_2}+\frac{1}{c_3}) < 1$, *then the control problem* (2.1)–(2.2) *has a unique solution* $(\alpha_1^*,\alpha_2^*,\alpha_3^*) \in \mathcal{U}$ *and has the form of* (6.1)–(6.6).

Proof By Lemma 7.1 and Ekeland's variational principle, we claim that, for each $\varepsilon > 0$, there exists $(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon) \in \mathcal{U}$ such that

$$\tilde{J}(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon) \ge \sup_{(\alpha_1,\alpha_2,\alpha_3)\in\mathcal{U}} \tilde{J}(\alpha_1,\alpha_2,\alpha_3) - \varepsilon,$$

$$\tilde{J}(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon) \ge \sup_{(\alpha_1,\alpha_2,\alpha_3)\in\mathcal{U}} \left\{ \begin{array}{l} \tilde{J}(\alpha_1,\alpha_2,\alpha_3) - \sqrt{\varepsilon}\|\alpha_1^\varepsilon - \alpha_1\|_{L^1(Q_T)} \\ -\sqrt{\varepsilon}\|\alpha_2^\varepsilon - \alpha_2\|_{L^1(Q_T)} - \sqrt{\varepsilon}\|\alpha_3^\varepsilon - \alpha_3\|_{L^1(0,T)} \end{array} \right\}.$$

Thus the perturbed functional

$$\begin{aligned} \tilde{J}_\varepsilon(\alpha_1,\alpha_2,\alpha_3) = \tilde{J}(\alpha_1,\alpha_2,\alpha_3) &- \sqrt{\varepsilon}\|\alpha_1^\varepsilon - \alpha_1\|_{L^1(Q_T)} \\ &- \sqrt{\varepsilon}\|\alpha_2^\varepsilon - \alpha_2\|_{L^1(Q_T)} - \sqrt{\varepsilon}\|\alpha_3^\varepsilon - \alpha_3\|_{L^1(0,T)} \end{aligned}$$

attains its supremum at $(\alpha_1^\varepsilon,\alpha_2^\varepsilon,\alpha_3^\varepsilon)$. Similar arguments as in those in the proof of Theorem 6.1 will produce

$$\begin{aligned} \alpha_1^\varepsilon(s,t) &= \mathcal{F}_1\left[\frac{[\omega_1(s,t)+\xi_1^\varepsilon(s,t)]p_1^\varepsilon(s,t)}{c_1} + \frac{\sqrt{\varepsilon}\theta_1(s,t)}{c_1}\right], \\ \alpha_2^\varepsilon(x,t) &= \mathcal{F}_2\left[\frac{[\omega_2(x,t)+\xi_2^\varepsilon(x,t)]p_2^\varepsilon(x,t)}{c_2} + \frac{\sqrt{\varepsilon}\theta_2(x,t)}{c_2}\right], \\ \alpha_3^\varepsilon(t) &= \mathcal{F}_3\left[\frac{[\omega_3(t)+\eta^\varepsilon(t)]q^\varepsilon(t)}{c_3} + \frac{\sqrt{\varepsilon}\theta_3(t)}{c_3}\right], \end{aligned}$$

where $\theta_1 \in L^\infty(Q_T)$, $\theta_2 \in L^\infty(Q_T)$, $\theta_3 \in L^\infty(0,T)$, and $|\theta_1(s,t)| \le 1$, $|\theta_2(x,t)| \le 1$, $|\theta_3(t)| \le 1$.

We first prove the uniqueness by using the contraction mapping principle. Define the mapping $\mathcal{C} : \mathcal{U} \to \mathcal{U}$ by

$$\begin{aligned} \mathcal{C}(\alpha_1,\alpha_2,\alpha_3) &= \mathcal{F}\left(\frac{(\omega_1+\xi_1)p_1}{c_1}, \frac{(\omega_2+\xi_2)p_2}{c_2}, \frac{(\omega_3+\eta)q}{c_3}\right) \\ &= \left(\mathcal{F}_1\left[\frac{(\omega_1+\xi_1)p_1}{c_1}\right], \mathcal{F}_2\left[\frac{(\omega_2+\xi_2)p}{c_2}\right], \mathcal{F}_3\left[\frac{(\omega_3+\eta)q}{c_3}\right]\right), \end{aligned}$$

where (p_1, p_2, q) and (ξ_1, ξ_2, η) are, respectively, solutions of the state system and the adjoint system corresponding to the control variable $(\alpha_1,\alpha_2,\alpha_3)$. From Theorems 4.1

and 5.2, we know that (p_1, p_2, q) and (ξ_1, ξ_1, η) are continuous with respect to the control variable $(\alpha_1, \alpha_2, \alpha_3)$. So we have

$$
\begin{aligned}
&\|\mathcal{C}(\alpha_1, \alpha_2, \alpha_3) - \mathcal{C}(\alpha_1', \alpha_2', \alpha_3')\| \\
&= \sum_{i=1}^{2} \left\| \mathcal{F}_i \left(\frac{[\omega_i + \xi_i] p_i}{c_i} \right) - \mathcal{F}_i \left(\frac{[\omega_i + \xi_i'] p_i'}{c_i} \right) \right\|_{L^\infty(Q_T)} \\
&\quad + \left\| \mathcal{F}_3 \left(\frac{[\omega_3 + \eta] q}{c_3} \right) - \mathcal{F}_3 \left(\frac{[\omega_3 + \eta'] q'}{c_3} \right) \right\|_{L^\infty(0,T)} \\
&\le \sum_{i=1}^{2} \left\| \frac{(\omega_i + \xi_i) p_i}{c_i} - \frac{(\omega_i + \xi_i') p_i'}{c_i} \right\|_{L^\infty(Q_T)} + \left\| \frac{(\omega_3 + \eta) q}{c_3} - \frac{(\omega_3 + \eta') q'}{c_3} \right\|_{L^\infty(0,T)} \\
&\le B_1 T \left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} \right) \left(\|\alpha_1 - \alpha_1'\|_{L^\infty(Q_T)} + \|\alpha_2 - \alpha_2'\|_{L^\infty(Q_T)} + \|\alpha_3 - \alpha_3'\|_{L^\infty(0,T)} \right),
\end{aligned}
$$

where B_1 is a positive constant. Since $B_1 T(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}) < 1$, the mapping $\mathcal{C}$ is a contraction and owns a unique fixed point $(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) \in \mathcal{U}$. Then Theorem 6.1 implies that any optimal control $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$ if exists must be a fixed point of the mapping $\mathcal{C}$. Hence the uniqueness of optimal policies holds.

Next, we show that the control $(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3)$ is the optimal control. Since

$$
\begin{aligned}
&\|\mathcal{C}(\alpha_1^\varepsilon, \alpha_2^\varepsilon, \alpha_3^\varepsilon) - (\alpha_1^\varepsilon, \alpha_2^\varepsilon, \alpha_3^\varepsilon)\|_\infty \\
&= \Bigg\| \left(\mathcal{F}_1 \left[\frac{(\omega_1 + \xi_1^\varepsilon) p_1^\varepsilon}{c_1} \right], \mathcal{F}_2 \left[\frac{(\omega_2 + \xi_2^\varepsilon) p_2^\varepsilon}{c_2} \right], \mathcal{F}_3 \left[\frac{(\omega_3 + \eta^\varepsilon) q^\varepsilon}{c_3} \right] \right) \\
&\quad - \left(\mathcal{F}_1 \left[\frac{[\omega_1 + \xi_1^\varepsilon] p_1^\varepsilon}{c_1} + \frac{\sqrt{\varepsilon}\theta_1}{c_1} \right], \mathcal{F}_2 \left[\frac{[\omega_2 + \xi_2^\varepsilon] p_2^\varepsilon}{c_2} + \frac{\sqrt{\varepsilon}\theta_2}{c_2} \right], \right. \\
&\quad \left. \mathcal{F}_3 \left[\frac{[\omega_3 + \eta^\varepsilon] q^\varepsilon}{c_3} + \frac{\sqrt{\varepsilon}\theta_3}{c_3} \right] \right) \Bigg\|_\infty \\
&= \sum_{i=1}^{2} \left\| \left(\mathcal{F}_i \left[\frac{(\omega_i + \xi_i^\varepsilon) p_i^\varepsilon}{c_i} \right] - \mathcal{F}_i \left[\frac{[\omega_i + \xi_i^\varepsilon] p_i^\varepsilon}{c_i} + \frac{\sqrt{\varepsilon}\theta_i}{c_i} \right] \right) \right\|_\infty \\
&\quad + \left\| \mathcal{F}_3 \left[\frac{(\omega_3 + \eta^\varepsilon) q^\varepsilon}{c_3} \right] - \mathcal{F}_3 \left[\frac{[\omega_3 + \eta^\varepsilon] q^\varepsilon}{c_3} + \frac{\sqrt{\varepsilon}\theta_3}{c_3} \right] \right\|_\infty \\
&\le \sum_{i=1}^{2} \left\| \frac{(\omega_i + \xi_i^\varepsilon) p_i^\varepsilon}{c_i} - \frac{(\omega_i + \xi_i^\varepsilon) p_i^\varepsilon}{c_i} - \frac{\sqrt{\varepsilon}\theta_i}{c_i} \right\|_\infty \\
&\quad + \left\| \frac{(\omega_3 + \eta^\varepsilon) q^\varepsilon}{c_3} - \frac{(\omega_3 + \eta^\varepsilon) q^\varepsilon}{c_3} - \frac{\sqrt{\varepsilon}\theta_3}{c_3} \right\|_\infty \\
&\le \frac{1}{c_1}\sqrt{\varepsilon}\|\theta_1(s,t)\|_\infty + \frac{1}{c_2}\sqrt{\varepsilon}\|\theta_2(x,t)\|_\infty + \frac{1}{c_3}\sqrt{\varepsilon}\|\theta_3(t)\|_\infty \\
&\le \sqrt{\varepsilon} \left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} \right),
\end{aligned}
$$

it is easy to show

$$
\begin{aligned}
&\|(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) - (\alpha_1^{\varepsilon}, \alpha_2^{\varepsilon}, \alpha_3^{\varepsilon})\|_{\infty} \\
&\quad = \|\mathcal{C}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) - (\alpha_1^{\varepsilon}, \alpha_2^{\varepsilon}, \alpha_3^{\varepsilon})\|_{\infty} \\
&\quad = \|\mathcal{C}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) - \mathcal{C}(\alpha_1^{\varepsilon}, \alpha_2^{\varepsilon}, \alpha_3^{\varepsilon}) + \mathcal{C}(\alpha_1^{\varepsilon}, \alpha_2^{\varepsilon}, \alpha_3^{\varepsilon}) - (\alpha_1^{\varepsilon}, \alpha_2^{\varepsilon}, \alpha_3^{\varepsilon})\|_{\infty} \\
&\quad \le \|\mathcal{C}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) - \mathcal{C}(\alpha_1^{\varepsilon}, \alpha_2^{\varepsilon}, \alpha_3^{\varepsilon})\|_{\infty} + \|\mathcal{C}(\alpha_1^{\varepsilon}, \alpha_2^{\varepsilon}, \alpha_3^{\varepsilon}) - (\alpha_1^{\varepsilon}, \alpha_2^{\varepsilon}, \alpha_3^{\varepsilon})\|_{\infty} \\
&\quad \le B_1 T \left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}\right) (\|\bar{\alpha}_1 - \alpha_1^{\varepsilon}\|_{\infty} + \|\bar{\alpha}_2 - \alpha_2^{\varepsilon} + \|\bar{\alpha}_3 - \alpha_3^{\varepsilon}\|_{\infty}) \\
&\qquad + \sqrt{\varepsilon}\left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}\right).
\end{aligned}
$$

Note that

$$
\|(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) - (\alpha_1^{\varepsilon}, \alpha_2^{\varepsilon}, \alpha_3^{\varepsilon})\|_{\infty} = \|\bar{\alpha}_1 - \alpha_1^{\varepsilon}\|_{\infty} + \|\bar{\alpha}_2 - \alpha_2^{\varepsilon}\|_{\infty} + \|\bar{\alpha}_3 - \alpha_3^{\varepsilon}\|_{\infty}.
$$

So we can get

$$
\begin{aligned}
&\|(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) - (\alpha_1^{\varepsilon}, \alpha_2^{\varepsilon}, \alpha_3^{\varepsilon})\|_{\infty} \\
&\quad \le B_1 T \left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}\right) \|(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) - (\alpha_1^{\varepsilon}, \alpha_2^{\varepsilon}, \alpha_3^{\varepsilon})\|_{\infty} \\
&\qquad + \sqrt{\varepsilon}\left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}\right),
\end{aligned}
$$

that is,

$$
\|(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) - (\alpha_1^{\varepsilon}, \alpha_2^{\varepsilon}, \alpha_3^{\varepsilon})\|_{\infty} \le \frac{\sqrt{\varepsilon}(c_1^{-1} + c_2^{-1} + c_3^{-1})}{1 - B_1 T (c_1^{-1} + c_2^{-1} + c_3^{-1})}.
$$

Therefore, $(\alpha_1^{\varepsilon}, \alpha_2^{\varepsilon}, \alpha_3^{\varepsilon}) \to (\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3)$ as $\varepsilon \to 0$. It follows from Lemma 7.1 that

$$
\tilde{J}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) = \sup_{(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{U}} \tilde{J}(\alpha_1, \alpha_2, \alpha_3),
$$

which implies that $(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) \in \mathcal{U}$ is the optimal policy and has the form of (6.1)–(6.6). □

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